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Experiencing Change: The Mathematics of Change in Multiple Environments

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In the SimCalc project and in the Mathematics of Change group at TERC, we are investigating how students from elementary through high school learn about the mathematics of change in multiple mathematical environments. As part of this research, we studied 5th-grade students doing mathematics-of-change activities from the *Investigations* curriculum (Russell, Tierney, Mokros, & Economopoulos, 1998) in multiple mathematical environments. This experience has led us to question the view that students connect experiences in different environments by recognizing a core mathematical structure that is common to all environments. We propose an alternative perspective on learning, in which students make mathematical environments into lived-in spaces for themselves and connect environments through the development of family resemblances across their experiences.

Key Words: Calculus/analysis; Children's strategies; Connections in mathematics; Elementary (K–8); Manipulatives; Patterns/relationships in mathematics; Representations/modeling

There is strong support in the mathematics education community for the view that students should encounter mathematical concepts in multiple mathematical environments (Behr, Lesh, Post, & Silver, 1983; Brenner, Herman, Ho, & Zimmer, 1999; Dienes, 1964; Janvier, 1987; Rubin, 1990; Yerushalmy & Schwartz, 1993). In American schools, fraction bars, counters, drawings, and symbols are often used for teaching students about fractions (Ball, 1992; Behr et al., 1983). Tables, graphs, equations, and sometimes physical devices are used when functions are studied (Confrey & Smith, 1991; Kaput, 1989; Solomon, Barros, et al., 1999; Solomon, Kelemanik, et al., 1999). Yet, teachers and researchers have legitimate concerns about confusing students with too unusual or numerous mathematical environments (Dufour-Janvier, Bednarz, & Belanger, 1987). How can these multiple environments be used to help students to learn mathematics? Where is the mathematics

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that students are to learn from any given environment, and how do students make connections among different environments?

A mathematically experienced adult may expect students to see the same mathematics in an activity that she herself, on the basis of long experience with the mathematics and the activity, can see. These expectations may lead one to believe that mathematical materials or activities in themselves can convey the mathematical ideas that students are expected to learn and that the core mathematical relationships that adults recognize as commonalities among different activities are the only mathematically relevant connections to be made among these activities. However, researchers have found that students often do not make the connections they are expected to make and sometimes do not even believe they should find similar results in different environments (Dufour-Janvier et al., 1987; Noble, Nemirovsky, Tierney, & Wright, 1999). We will suggest an alternative view of learning in multiple environments that takes into account students' ways to inhabit and make sense of each environment. We will also articulate a view of the process of making connections among experiences in different mathematical environments; this view allows for diverse and numerous connections among experiences.

In the first part of this article, we address the question of where the mathematics that we expect students to learn is located. We trace the origins of some common assumptions and offer an alternative view of where the mathematics is located; this view will later be grounded in the analysis of 2 fifth-grade students' experiences of working with multiple mathematical environments. In this view, the mathematics that students learn from working in a given environment emerges from their process of making that environment into a lived-in space for themselves (Nemirovsky, Tierney, & Wright, 1998).

In the second part of the article, we look at the ways in which students make connections among their experiences in multiple lived-in spaces. We propose that Wittgenstein's idea of "family resemblances" (1958, p. 32) can be used to describe the kinds of commonalities students come to see among their experiences in multiple environments.

WHERE IS THE MATHEMATICS?

In the work of Dienes (1964), we find the clear articulation of some ideas, implicit in the work of other researchers, about where the mathematics that children are to learn is located. Dienes used the term *embodiment* to describe a problem situation in which a mathematical structure can be found. Dienes argued that students should be presented with many embodiments of a mathematical structure to "ensure that eventually only the essentially mathematical structure is retained out of all the embodied situations" (p. 41). In this description, the variation of embodiments makes the essential mathematical structure visible, as an invariant. Dienes described the differences among embodiments as the "dressing up," emphasizing that for any embodiment, "the mathematical relevance lies in the number relationships in the problems, not in the 'dressing up'" (p. 35).

Yet Dienes clearly valued more than just mathematical structures as represented in symbolic form. Dienes emphasized the importance of imagery and storytelling in mathematics, stating that "symbol-manipulation in mathematics is all too often meaningless simply because there is no corresponding transformation of images" (1964, p. 105), and he developed a large number of games and stories for making advanced mathematical topics accessible to children.

Dienes's use of the term *embodiment* may contribute to some educators' assumptions that the mathematics students are to learn can be found in a manipulative material or other tool. A number of researchers have critiqued the idea that concrete manipulatives, physical devices, or computer software can, in themselves, embody mathematical ideas for a student (Ball, 1992; Cobb, Yackel, & Wood, 1992; Lesh, Post, & Behr, 1987a; Meira, 1998; Teasley & Roschelle, 1993). Meira criticized the view that specific physical devices can be more or less "transparent" to students in allowing access to mathematical meaning: "The transparency of devices follows from the very process of using them. That is, the transparency of a device emerges anew in every specific context and is created during activity through specific forms of using the device" (p. 138). Physical materials or computer software in the classroom may provide new opportunities for learning, but they do not insure that students will learn the intended mathematics.

Lesh et al. (1987a) have argued that Dienes's work has been misinterpreted by those who believe that Dienes stated that the mathematics resides in physical materials or computer software. Lesh et al. argued that Dienes meant that children abstract mathematical relations from their own *activities*—from their use of materials, not from the materials alone. This rephrased claim may in turn indicate that although the mathematics does not reside in the materials, it can be conveyed to all students through tightly scripted classroom activities involving the materials. However, researchers who have looked carefully at what students do in classroom or interview settings when given a prescribed task have found that the ways students act and make sense of their actions can vary widely from what is expected by the designer of the activity (Ball, 1990, 1992; Cobb et al., 1992; Meira, 1998; Nemirovsky, Noble, & Wright, 1999; Noble et al., 1999; Teasley & Roschelle, 1993). This work indicates that mathematical concepts reside not in physical materials, computer software, or prescribed classroom activities but in what students do and experience.

In this article, we use the word *environment* to describe a configuration of tools (such as manipulative materials, a written number table, or computer software), tasks, expectations, conventions, and rules for tool use negotiated in the classroom. We use this term to emphasize that mathematical environments are "'thinking spaces' for working on ideas" (Ball, 1990, p. 7); they do not contain or represent the mathematics to be learned but may provide opportunities for students to develop mathematical ideas. Of course some environments may provide more opportunities than others for students to work with some set of mathematical ideas, but it is up to the learner to inhabit a mathematical environment and to develop the mathematics she or he will learn through work in the environment.

When a student inhabits a mathematical environment, populating it with her or his actions, intentions, and interactions with others, we say that the student has made the environment a *lived-in space* for herself or himself. Nemirovsky et al. (1998) have articulated the notion of lived-in spaces in the context of the experiences of two girls learning about graphing through the use of a motion detector. Nemirovsky et al. described the way these two girls made the graphs of position versus time produced on a computer screen into "lived-in spaces populated by specific intentions (e.g., keeping the [graphical] mountains the same height, separating the graphs, creating a flat line) and different types of actions (e.g., hand separation, getting closer, leaving the train straight)" (p. 163). These researchers described lived-in spaces as "relational, intentional, and creative" (p. 153). If a space is *relational,* "changes, even if they are physically circumscribed to a particular aspect [of a lived-in space], affect the lived-in space as a whole" (p. 153). A space is *intentional* if it is a place to "do things and accomplish purposes" (p. 153), and it is *creative* if it is constantly being recreated as it is experienced. If the purpose of a mathematical activity is not simply to learn the rules and conventions of an environment but instead is to make the environment a lived-in space for oneself, then a diverse range of actions and intentions may be legitimate parts of that mathematical activity.

In our work, we attempt to understand how mathematical environments can become lived-in spaces for students and how this process relates to students' developing mathematical understandings. We will use an analogy between a dwelling and our understanding of a lived-in space. What is significant about living in a dwelling that has several rooms? Each room has its own uses, and what is suitable to do in the kitchen is often inappropriate to do in a bedroom, and vice versa. One is never just in the dwelling but is always in this or in that room of the dwelling. Our familiarity with the dwelling grows and develops as we act according to our own purposes and expectations in the various rooms of the dwelling. The similarities and differences among these rooms emerge from our experiences, and both similarities and differences are critical to what the dwelling comes to mean for us. We trace an analogy between this imaginary dwelling and the mathematics of change. There is no ultimate core principle that encapsulates all that this mathematics is, in the same way that there is no single corner of the dwelling that tells us everything about the dwelling, in part because experiencing that corner cannot necessarily tell us how we will experience another corner.[1]

This image of mathematical understanding may seem odd to some readers, especially because a mathematical domain is said to have achieved maturity when all its contents can be derived from a small set of axioms. No achievement in the history of mathematics is more celebrated than the formulation of the Euclidean geometry as entirely derived from five axioms. However, we distinguish between

[1]Of course, not all corners of a dwelling (or properties or postulates of a mathematical system) are of equal importance. Given the purposes at hand (trying to find the basic axioms is a particular purpose), some corners are more useful than others.

the structure of a mathematical system and the nature of mathematical understanding. Although the derivation of a system from a set of axioms can be an important part of learning mathematics, mathematical understanding arises through numerous and varied experiences that allow one to make mathematical environments into lived-in spaces for oneself. Each environment engages one with some aspects of the mathematics and not with others, not only because of the attributes of any one environment but also because of the intentions, expectations, and actions one brings to each environment.

HOW DO STUDENTS MAKE CONNECTIONS AMONG ENVIRONMENTS?

If students learn mathematics by making mathematical environments into lived-in spaces for themselves, then how do students connect their experiences in different environments? We will begin to explore this question by referring to the work of mathematics educators who study students' use of multiple representations. The term *representations* has often been used to describe tools for representing mathematical ideas, especially when these tools are number tables, Cartesian graphs, and equations (Confrey, 1990). Among these researchers, Behr et al. (1983), by stating that "[a] major hypothesis of the [Rational Number] project is that it is the ability to make translations among and within these several modes of representation that makes ideas meaningful to learners" (p. 102), have emphasized the importance of moving among and connecting multiple representations. Researchers have also found that differences among representations are important for students' learning (Confrey & Smith, 1991; Kaput, 1987; Lesh, Post, & Behr, 1987b; Moschkovich, Schoenfeld, & Arcavi, 1993). Confrey and Smith (1991) have argued that "each representation yields its own insights into functions such that no one can be subordinated to another" (p. 59). This work indicates not only that multiple representations are valuable for learning mathematics but also that the differences among these representations can contribute to this value.

To further explore the ways that similarities and differences among multiple environments influence students' learning of mathematics, we propose a perspective in which students learn mathematics through the development of "family resemblances" (Wittgenstein, 1958, p. 32) across lived-in spaces. In his account of the use of words, Wittgenstein argued that words are used not by "applying" definitions but by noticing "family resemblances" (p. 32) among differing uses of a word. Wittgenstein explored the meaning of the word *game* as it is used to describe ball games, word games, card games, and so on. Wittgenstein argued that if you look at all the "proceedings that we call 'games', … you will not see something that is common to *all*, but similarities, relationships, and a whole series of them at that" (p. 31). He described these similarities and relationships as family resemblances because they are like the resemblances among members of a family, any two of whom have some characteristic of height, shape, gait, and so on in common but who rarely all share a single characteristic. Thus, according to Wittgenstein,

"'Games' form a family" in the sense of having a "complicated network of similarities overlapping and criss-crossing" (p. 32).

Wittgenstein (1958) himself related this idea to mathematical concepts, describing the meaning of the word *number* as follows:

> Why do we call something a "number"? Well, perhaps because it has a direct relationship with several things that have hitherto been called number; and this can be said to give an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (p. 32)

Thus, Wittgenstein argued that there is no unique or ultimate description that accounts for the use of a word such as *number,* only a partial and changing likeness developed through a web of contexts.[2] Being a proficient user of the concept of number is being able to move fluently through extended networks of partial similarities and differences.

In this article, we describe the ways in which, through the development of family resemblances across lived-in spaces, two students achieved a certain fluency in some aspects of the mathematics of change. Instead of finding one common thread through all their experiences in different environments, these students found a "complicated network of similarities overlapping and criss-crossing" (Wittgenstein, 1958, p. 32). For these students, the differences among their experiences of environments as lived-in spaces contributed as much to their understanding of the nature of the mathematics of change as did the commonalities among the experiences.

THE MATHEMATICS OF CHANGE

Mathematics educators have named the mathematics of change as an important strand of school mathematics (National Council of Teachers of Mathematics, 1989, 2000; Stewart, 1990), not only because of its critical role historically and in the present day in mathematics, the sciences, and the social sciences but also because the concepts of the mathematics of change are rooted in everyday experiences that young children can understand (Confrey, 1995; Kaput & Nemirovsky, 1995; Nemirovsky, 1993).

The mathematics of change can be used to describe the way a plant grows or how one catches up to a friend by running down the street. Two fundamental concepts of the mathematics of change are *rate of change* and *accumulation*. These quantities and their relationship constitute much of the subject matter of calculus. Rate of

[2]This idea is related to Dienes's discussion of the formation of *categories* in mathematics:

> "Four" is not a thing at all, so we cannot ever come across it in any physical sense. We can only come across four suitcases, four children, any four objects, or any four entities whose separateness from each other is clearly understood. The concept of "four" is the result of a number of experiences that finally led to the formation and the use of a *category.* (1969, p. xvii)

change can be recognized in calculus class as the derivative. Rate of change can also be recognized in algebra classes concerned with the slopes of lines or the successive differences or ratios in a table of values, which can show change over time for functions (Confrey & Smith, 1995). Rates of change can also be part of learning about rational numbers in presecondary school mathematics (Behr et al., 1983).

Among mathematics educators, ideas about what a rate of change is are almost as numerous as the contexts for exploring rate of change. Rate of change can be seen as a directly perceived quantity, such as the speed of your own walking, or as a mathematical relationship of two quantities, such as distance and time, that yields a new quantity, such as speed. Many descriptions of rate of change incorporate aspects of both these views, recognizing the direct perceptual experience of rate of change as well as the numerical relationship it reflects (Behr et al., 1983; Confrey & Smith, 1995; Monk & Nemirovsky, 1994; Piaget, 1946/1970; Thompson, 1994).

Accumulation arises in calculus as the integral, often introduced as the area under a curve. Accumulation is linked to the concept of rate because the changes specified by the rate of change of a quantity over time add up to yield the accumulated change in that quantity. The concept of accumulation can also be introduced to students from elementary through high school in a variety of contexts (Confrey, 1992; Confrey & Smith, 1995; Kaput & Nemirovsky, 1995; Nemirovsky, Tierney, & Ogonowski, 1993). For instance, the speed, or rate of change, as measured on the speedometer of a car also tells the driver how fast she is accumulating distance traveled. Thus one traveling at a speed of 60 mph (96 kph) is accumulating distance at twice the rate of one traveling at 30 mph (48 kph). This fundamental relationship between rate of change and accumulation can be seen in this example as well as in the relationship between the derivative and the integral in calculus.

The curriculum unit used in this study included many activities designed to familiarize students with the mathematics of change in the context of motion and, especially, the relationship between step-size and distance traveled—a discrete version of rate of change and accumulation.

THE STUDY

The fifth-grade *Investigations* (Russell et al., 1998) mathematics of change unit, “Patterns of Change: Tables and Graphs” (Tierney, Nemirovsky, Noble, & Clements, 1998), was pilot-tested in two classrooms, one in Cambridge and one in Boston. This article concerns our research in the Boston classroom. The classroom teachers who pilot-tested the materials met regularly with the curriculum developers to collaboratively plan the testing, revise the curriculum, and analyze the data collected during the testing. In addition to providing feedback for the curriculum developers, the pilot tests also provided us with an opportunity to conduct research on student learning about the mathematics of change.

We took the opportunity to film classroom activities on the mathematics of change while this curriculum was being pilot-tested. In our research study, we

focused on understanding how students made sense of ideas of mathematics of change in a classroom context and, more specifically, how work with number tables, physical materials, and computer simulations contributed to students' understandings of the mathematics of change. To explore detailed cases of students' work in a classroom context, we filmed the whole classroom during whole-class activities and chose one group of students as a focus group to film consistently during small-group work. Two researchers were in the classroom for each day of the study, one to film classroom activities and the other to interact with students to elicit their ideas about the work they were doing. The focus group consisted of four students, two boys and two girls, who were chosen by the classroom teacher as representative of the range of mathematical abilities found in her classroom. This group often worked in pairs during small-group work, and the video camera alternated between the two pairs during such periods. The students in the focus group were also interviewed individually and as a group at regular intervals during the course of the 4-week unit; copies of their written work were collected throughout the unit.

We reviewed all the data taken in the two pilot tests and transcribed and analyzed many selected classroom and interview sessions. We used an interpretive approach (Packer & Mergendoller, 1989) in our analysis, focusing on the details and meanings of the actions and utterances of the students, teacher, and researcher in the classroom. We treated the participants' utterances and actions as processes accomplished over time and in interaction with others "rather than [as] born naturally whole out of the speaker's forehead, the delivery of a cognitive plan" (Schegloff, 1982, p. 73). We chose video episodes to explore in greater detail by selecting episodes that were puzzling to us in some way or that showed students working on significant mathematical issues in uncommon ways. We chose for this case study one such episode, in which a pair of boys from the focus group used physical motions, a number table, and a computer simulation to analyze linear motion.

THE EPISODES

We present three episodes of the joint work of 2 fifth-grade boys, Norman and Luke, from our focus group on one day of class approximately halfway through the 4-week pilot version of "Patterns of Change: Tables and Graphs" (Tierney et al., 1998). Ms. P. is the teacher in this classroom, and Tracey Wright is a researcher whose role on this day of class was to interact with students while another researcher filmed classroom activities. Prior to Episode 1, students had carried out an investigation of speed and position as they took "trips" by walking along straight lines of masking tape on the classroom floor. Students dropped bean bags at regular time intervals and then discussed the relationship of the resulting trails of bean bags to the motion of the person who had dropped them; this activity eventually led to students' recording position and speed data in tables and graphs. Episode 1 occurred at the beginning of this unit's next investigation, in which students began exploring other ways of taking trips.

Episode 1: Cuisenaire-Rods-and-Meterstick Environment

Introduction. In Episode 1, the two boys used Cuisenaire rods of two sizes to take "steps" by flipping the rods end over end, moving in parallel along the two sides of a meterstick, using the markings on the meterstick as a measure of the progress of each Cuisenaire rod.

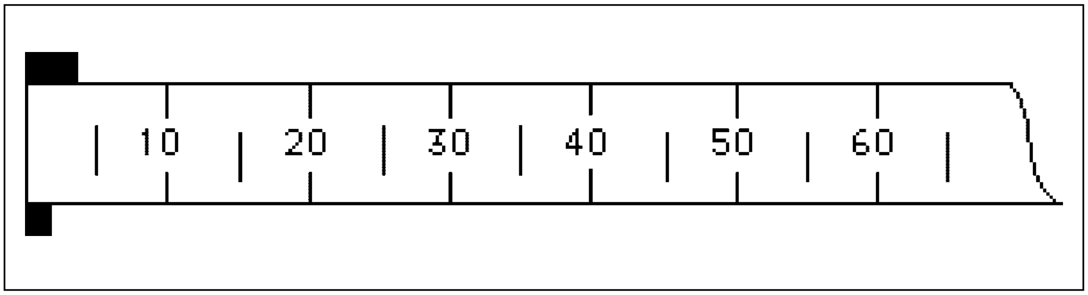

Figure 1. Meterstick and Cuisenaire rods after 1 step.

In the ideal case, a student takes a step of 2 cm at the same time his partner takes a step of 4 cm, and thus the rate of change in position of the second student's rod is twice that of the first student's. One can make predictions about where each student's rod will be after a given number of steps; for instance, after 10 steps, the students' rods will be at 20 cm and 40 cm, respectively.

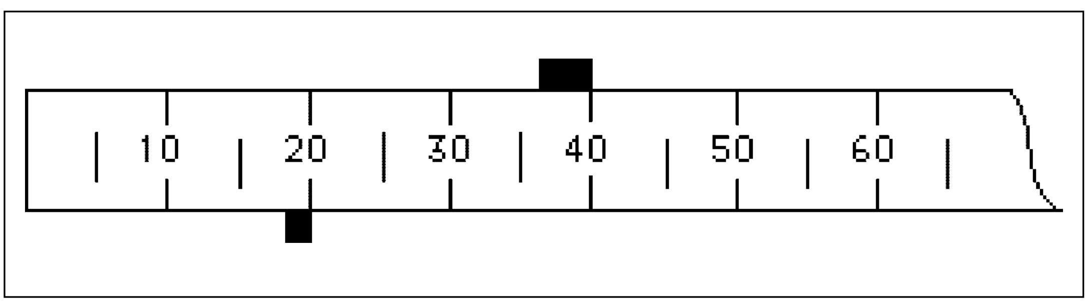

Figure 2. Meterstick and Cuisenaire rods after 10 steps.

In fact, one would predict that the student with the 2-cm rod would always be only half as far along the meterstick as the student with the 4-cm rod. Thus one can describe the rates of change of position for the two Cuisenaire rods and the relationships of the rates of change to the accumulated distances traveled. A trip ends when either boy reaches the end of the meterstick (100 cm) with his Cuisenaire rod.

Annotated transcript. In this class session, Norman and Luke are asked to enact a trip using Cuisenaire rods and a meterstick, to satisfy a motion story from the curriculum: "The girl gets to the tree way ahead of the boy." Norman chooses the role of the girl and Luke the role of the boy, and the far end of the meterstick is identified with the tree marking the end of the trip. Norman and Luke choose the 4-cm and 2-cm Cuisenaire rods, respectively, which they will move along opposite sides of the meterstick, to take steps of the same size as the rods. Ms. P. tries to get the boys to predict where Luke will be when Norman reaches the end of the meterstick, and the boys respond:

Luke:	He'll be around like over there, and I'll be up around there [pointing to places on the meterstick that are not visible in the videotape].
Ms. P.:	Oh, no, not "like around." Where *will* you be?
Luke:	I'll be at the 90, or 98.

The boys struggle to figure out how to answer this question, and finally Norman answers.

Norman:	Ninety-two, about. Come on, let's start the race off! We've got to start our … [they place their rods at the beginning to start]. Ready, set, go.
Luke:	[Calling out his positions as he slides his rod along the meterstick] 1, 2, 4, 6, 8, 10, 12.…
Norman:	Geeze, I'm at the 16 [as he moves his rod to the 16-cm mark].
Together: Norman:	Twenty, 24, 28. Geeze, the thing keeps slipping off [the rod slips on the table as he flips it end over end to take steps].
Luke:	Fourteen, 16, 18.…

The boys resisted figuring out the exact endpoints of the race; instead of guessing the outcome ahead of time, Norman chose to "start the race off!" to find out what would happen. In writing the curriculum, we had avoided using the language of racing because we saw the trips as having predetermined outcomes. However, given many children's familiarity with racing, we were not surprised that Norman connected his experience of racing to this activity. The boys' movement along the meterstick proceeded in a less predictable fashion than we had expected, because they were not taking steps at the same time. When Luke reached 16, Norman had reached only 24, not the value of 32 that one would expect, given that Norman's Cuisenaire rod was twice the length of Luke's.

> Norman and Luke continue with their trip, with Norman continuing to have difficulties with moving his rod and calculating to where he should move next while Luke continues to move easily along, counting by 2s. Eventually, Norman reaches the end of the meterstick with his Cuisenaire rod.

Norman:	I got to 100.
Luke:	Seventy-eight.…

Norman: [To Luke] You are at 76. I got.…

Luke: Seventy-six?

Norman: Yeah. Miss P., Miss P.!

Luke: I got around seven-….

Ms. P.: [From across the room] Uh, wait, I am just helping other kids for a minute.

Tracey: What happened? You got to [inaudible].

Norman: I got to 100 and he got to 76.

Tracey: Huh. And you [Luke] were moving in steps of 2.…

Luke: Two, and he was moving in steps of 4 [Norman holds up the 4-cm Cuisenaire rod and then four fingers to show Tracey].

The boys' trip worked out differently than we had expected. Had they taken synchronized, accurate steps using a rod of length 2 and a rod of length 4, one would expect that Norman would reach 100 cm when Luke was at 50 cm. However, although Luke was able to count easily by 2s and thus determine where to move his 2-cm rod at each step, Norman was flipping his 4-cm rod end over end and then trying to compute the result of the previous number plus 4 to check that he had moved to the right number. Both tasks were challenging for Norman. The meterstick slipped around on the desks; Norman's rod slipped backward after he flipped it; Norman had to reach under Luke's arm to move his rod to the end of the track; and Norman had difficulties in counting, particularly across decades (e.g., 48 to 52). At the same time, Luke just kept counting by 2s and moving his Cuisenaire rod along to each number, toward the end of the meterstick. All these factors helped lead to the end result of 76 and 100, instead of the expected 50 and 100, for the boys' endpoints.

Norman begins immediately to talk to Tracey about what had happened.

Norman: But I could have probably have went even faster because I, I really, he probably, really, would have been at 72, probably. You know why? [*Luke:* Yeah.] Because I was trying to think about 48, 49, [*Luke:* Yeah.] 50, 51, 52 [holding up a finger for each count except 48], so.… 'Cause I couldn't … [shakes fingers next to his head, indicating his trouble calculating the answer], and then I hopped and he was still going.

Norman and Tracey continue to discuss this trip and how to improve future trips. Norman asks for tape and secures the meterstick to the desks in preparation for future trips. When Ms. P. joins the boys, Luke explains the trip to her.

Luke: He defeat me because, lookit, I was at 2 [placing the 2-cm rod at 2 cm on the meterstick] and we were talking about 2s. I was counting by 2s: 2, 4, [moves 2-cm rod to 4 cm] and then he came around to the 8 [points to 8 cm], and I came around to the 6 [moves rod to 6 cm], and he came around to the 12 [points to 12 cm].

Norman said that he "could have … went even faster," meaning that he could have been farther ahead of Luke than he was, indicating that he realized that his difficulties influenced the outcome of the trip and realized how they influenced it. When Luke described the trip (e.g., "I came around to the 6 and he came around to the 12"), he described his and Norman's moving the Cuisenaire rods in time with each other, which indicates that he had some idea that they were supposed to move together and perhaps did not notice that they were not doing so. Neither boy mentioned that the outcome of the trip should have been Norman—100, Luke—50.

Comments. Believing he could have gone "even faster," Norman described the trip as a personal struggle with calculating the step he should take from 48 to 52 while Luke "was still going," continuing to take steps while Norman struggled. In Norman's description of the trip, one can see his intense involvement with the details of making correct steps and his awareness, like that of a runner in a race, of his opponent's continuing to make progress while Norman himself was "stuck." When Luke described the trip, he described what was significant to him: that Norman had defeated him and that Norman was always ahead of him. The boys did not see the trip as a way to enact a predetermined mathematical pattern of adding 2s and 4s, and, in fact, Norman's difficulties obscured the pattern the boys were expected to enact. Instead, the boys saw the trip as a race—Norman said, "Come on, let's start the race off!"—that each of the boys was invested in winning. This view of the trip could be considered a problem, because the language of racing is not consistent with investigating a predetermined mathematical pattern. However, the language of racing allowed the boys to make the Cuisenaire-rods-and-meterstick environment into an intentional and creative lived-in space, as they made it into a place to run races, try to win, and find the outcome of the race.

Episode 2: The Table Environment

Introduction. In this episode, Ms. P. attempted to clarify with the boys what should have happened in the Cuisenaire-rods trip by beginning a simple written table of position values to represent an ideal trip, with Luke's positions in the left column and Norman's positions in the right column.

Annotated transcript

Ms. P.: [She gets a blank piece of paper and a pencil and writes 2, 4 at the top of the paper] If you [Luke] are going 2 and he [Norman] is going 4, then you [Luke] will be at?

Luke: Two.

Norman: Four.

Luke: I mean 4.

Ms. P.: Unless you got stuck on the ground. And he's going to be?

Luke: Eight.

Ms. P. fills in 4 and 8 in table form, as shown below. The table has no headings or labels.

2	4
4	8

Ms. P.: And then what's gonna happen?

Together: Luke: It's, it's like….

Norman: Six, 12 [pointing to the table].

Norman: Oh, if we did it by numbers, we'll probably see. But that'd be a long time.

Ms. P.: You might probably see if you do it by numbers.

Norman: We can try it? On the paper? [Ms. P. drops the pencil in front of him. Norman picks it up and turns the paper around and starts filling in the table.]

Ms. P.: Sure you can! Why not?

Luke: [Pointing to the two columns] Like 2, 2; 2 times 2 is 4; 4 times. They're doubling up.

Ms. P.: And what is he [Norman] doing, or she? Or whichever one is.… It's she [Norman has the role of the girl for purposes of enacting motion stories about a boy and a girl taking a trip].

Luke: He's quadrupling up.

Ms. P.: So do you see some number [*Luke:* Yeah.] patterns happening?

Luke: Yeah.

Norman was interested in making the table on paper, saying, "Oh, if I did it by numbers, we'll probably see." The *it* to which Norman referred may have been a trip that he wished to do "by numbers" to see how the trip would go without the difficulties of Episode 1.

As Norman completes the part of the table shown below, the boys discuss the table with Ms. P.

Table at this point

2	4
4	8
6	12
8	16
10	20

Norman: [He has been filling in the table on paper.] It's like, it's like even numbers [points to paper], like when you go through 2.

Ms. P.: This [number table] happens to be even numbers. Why?

Norman: Like when, like I'm ahead of him like the, like 2, 4, 6, 8 [counting by 2s on his fingers] like 4, 8; and then [points to the table] 6, 12; 8, 16; 10, 20 (smiles). It sorta seems … [continues filling in table].

Ms. P.: Keep going. See if that follows. Does it?

Luke: Twenty-four; I'll be at the 12.

Norman: Yep [filling in table].

Luke: Yeah. It's like a half … of, of something. It's like the half of something.

Tracey: What's half of something?

Luke: Yeah. Like this, like 2 and 4, 4 and 8, 8 and 16 [in sing-song voice].

The boys began to notice numerical relationships between the two sets of position values. Luke noticed the one-half relationship of each pair of positions, and Norman noticed even numbers in the table. The boys also described the table with statements about their own actions: "Like when, like *I'm* [italics added] ahead of him"; "24, *I'll be* [italics added] at the 12." The boys projected themselves and their experiences with the Cuisenaire rods into the table.

Norman finishes filling in the table, with some help from Luke and from a student teacher, who points out an error in one of his calculations. Norman finds that the trip ends with Luke's position column reading 50 and his own reading 102, because of another calculation error. When Tracey asks him about the ending values, he decides to just change the 102 to 100. Then Norman makes an announcement:

Norman: I beat him by 50. [*Tracey:* So.…] I was beating him by 50 the whole time!

Tracey: Yeah? The whole time?

Norman: [He points to the last 2 values in the table, 50, 100.] Right here [at 50, 100] it started beating him by 50.

Norman tries to do some calculations on the later values in the table to determine where he began beating Luke by 50, but he becomes frustrated with the calculations and abandons the topic. The boys discuss their table further with Tracey and with Ms. P., and they record their results on a student sheet provided as part of the unit. Then Ms. P. tells the class that it is time to switch stations, and the boys move to the computer.

When Norman ended the table with the values 50 and 100, he found an outcome for this trip different from the one the boys had found for the Cuisenaire-rods trip. The table method lacked the physical and timing difficulties that affected the outcome of the previous trip, so that the boys were able to find the correct outcome more easily. Norman announced this outcome first as his own action in relation to Luke—"*I beat him* [italics added] by 50"—and then, when he looked at the number relationships, as the action of some unidentified other: "Right here *it* [italics added] started beating him by 50."

Comments. In this episode, the boys focused on patterns in the numbers in the table, making salient such aspects as the evenness of all the numbers and the "half" relationship between the position values on the two columns. Even though the

endpoints in the table (50, 100) were different from the endpoints they had reached with the Cuisenaire rods (76, 100), Norman and Luke were able to connect their experiences in the two environments. They connected their experiences by projecting themselves and their own actions into the number table, in examples such as Norman's saying "like I'm ahead of him the, like 2, 4, 6, 8"; Luke's comment "24, I'll be at the 12"; and Norman's "I beat him by 50." In these examples, the boys fused their personal experiences of the Cuisenaire-rods environment and their past experiences of racing to describe the numbers in the number table in terms of their own actions. Nemirovsky et al. (1998) described *fusion* as talking, gesturing, and acting without distinguishing symbols from their referents. For example, Luke's utterance "I'll be at the 12" describes at once the number 12 in the number table and the interpretation of that number as his own position (or his Cuisenaire rod's position) on a linear path.

Using the table, the boys were able to get the outcome we had expected (50, 100), in part because, in the table, steps are automatically synchronized as long as the two values in each row are aligned. In this way, synchrony in time is easier to maintain in this activity than in the Cuisenaire-rods activity, because time in the table is an implicit index of the rows of the table.

Episode 3: The Trips© Software Environment

Introduction. In Episode 3, Norman and Luke explored the same kind of trip using the Trips© (Clements, Nemirovsky, & Sarama, 1995) software environment, in which computer-generated icons of a boy and a girl run along a linear 100-unit track, in accord with starting positions and step sizes that are specified by the user. The trip taken by the boy and girl icons can be analyzed by watching the movement of the icons themselves, their final position values, and the continuously updated display of the current time and the position of each character. As in the other environments, a trip in the software environment ends when either icon reaches the end of the 100-unit track. At the end of a trip, a table and a graph of positions over time for the boy and the girl are also available.

Annotated transcript. Norman and Luke decide to try out on the computer the same trip they had made in the Cuisenaire-rods and table environments. So, they assign the boy and girl characters step sizes of 2 and 4, respectively. They run the simulation, causing the characters to move across the screen, taking one step per second, achieving the outcome shown in Figure 3.

Luke: Yes, we were right. We *were* right.

Tracey: You were?

Norman: [Yelling across the room] Ms. P., we were right!

Tracey: What were you right about?

Norman: 'Cause remember when the paper said 50 to 100?

Tracey: Yeah.

Norman: It just happened.

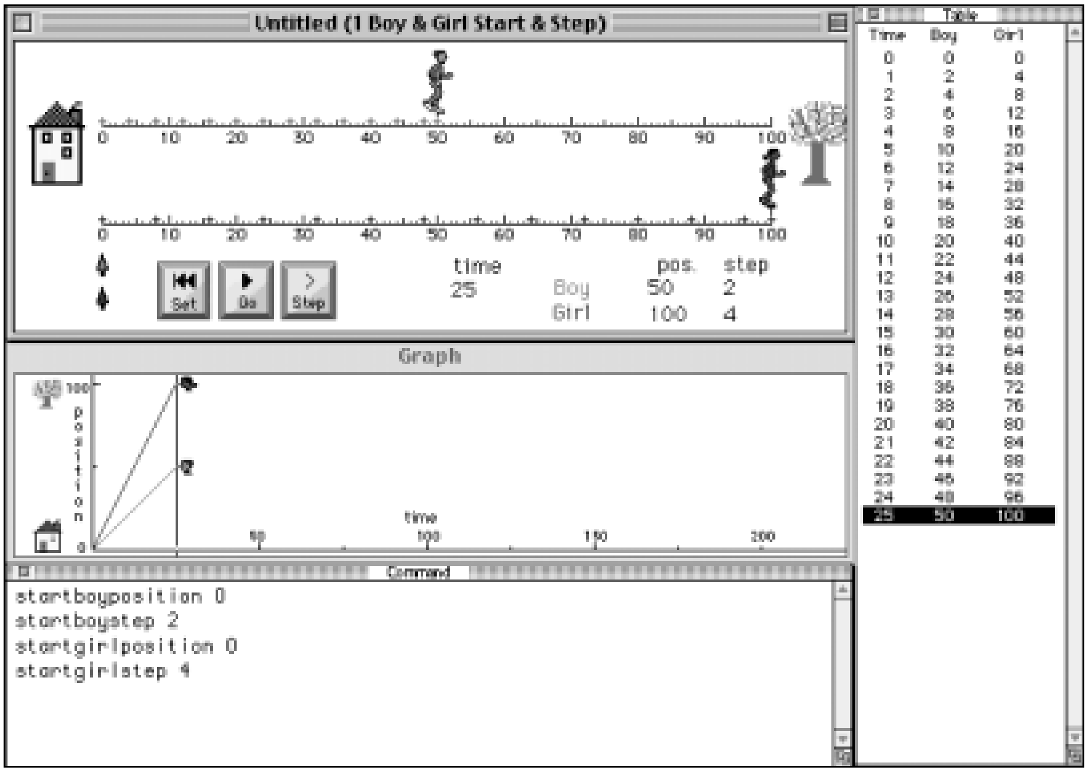

Figure 3. The Trips software screen after the first trip is completed.

Why were the boys so excited? The computer confirmed the result that Norman had found by making the table on paper. The boys seemed to recognize the sameness of the trips they had made with the table and with the computer and to see the computer as confirming their written work.

After a few minutes spent exploring the software and the trip he and Luke had just completed, Norman finds and uses a feature of the software that allows him to step through the trip, moving the boy and girl icons one step (and one second) at a time, by pressing a button on the screen, as shown in Figure 4.

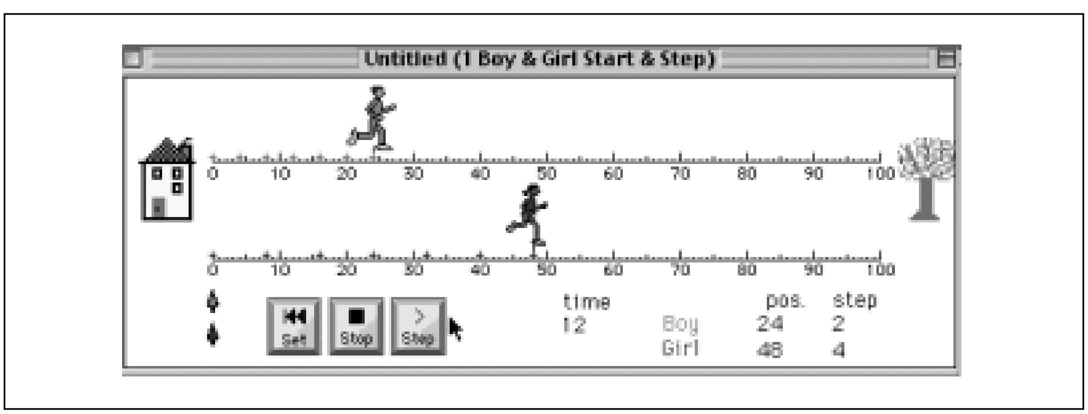

Figure 4. Pressing the "step" button allows one to step through the trip.

Tracey: You could click on stop if you don't want to step through it.

Norman: No, we were looking through the time [*Luke:* Keep on] [Norman continues stepping through the trip]. Ms. P., in 25 seconds, I was at 100.

The step-through feature allowed Norman to slow down the action of the trip and to initiate each step himself, just as he had taken his own steps with the Cuisenaire rod in Episode 1. Even though he was watching the characters of the boy and girl on the computer screen, Norman identified himself as the actor when he announced, "Ms. P., in 25 seconds, I was at 100." Stepping through the trip seems to have given Norman a sense of the total time the trip took to unfold and allowed him to make a connection to his experience of himself as an actor in the Cuisenaire-rods-and-meterstick trip.

Ms. P. joins the boys and asks them about the trip they have completed.

Ms. P: And did it work for you?

Luke: Yeah. 'Cause, look, it was a half [pointing to ending places of runners on screen].

Norman: He was at 50 and I was at a hundred. And on the paper, he was at 50 and I was on a hundred.

Comments. In this episode, the boys noticed that the trip in the software and the number-table trip had a common outcome: (50, 100); they announced, "We were right!" when the boy and girl icons reached the end of the trip in the software environment. When Ms. P. asked the boys if the trip worked for them, they pointed to the numerical patterns that the software trip had in common with the number-table trip. Luke pointed out that the endpoint for the boy was half that for the girl, which is the same relationship he had noticed in the number table on paper, and Norman noticed the common outcomes of the two trips: "He was at 50 and I was at a hundred [in the software]. And on the paper, he was at 50 and I was on a hundred." Norman also projected himself and Luke into the software environment in this last statement, using *He* and *I* to describe the actions of the icons in the software environment.

Norman was interested in "looking through the time" in this episode and in pursuing a detailed analysis of the steps in the trip by stepping through the trip one step (and one second) at a time. Stepping through the time in the software involved pressing the step button in the action window, which causes the boy and girl characters on the screen to advance by one step and causes the corresponding row of the table to be highlighted. Time can be rerun using this feature, allowing events to be broken down and reexamined and making time manipulable in a way that was not available in the other environments, in which time was an index or ordering of steps or number-table rows. Not surprisingly, Norman mentioned the duration of a trip for the first time while using the software, saying, "in 25 seconds, I was at 100."

WHERE IS THE MATHEMATICS?

Cuisenaire-Rods-and-Meterstick Environment

While telling stories of their trips, the boys repeatedly made sense of each new experience in terms of something familiar, although they did not make sense of these experiences in the ways we had expected. We, along with the boys' teacher, Ms. P., had expected that the boys might be able to use their knowledge of the two step sizes, 2 and 4, to predict something about the outcome of their trip before taking it. However, the boys were not engaged by this predicting task and instead jumped into the activity of taking a trip, describing it as racing. In this way, Norman and Luke made the Cuisenaire-rods-and-meterstick environment into an intentional and creative lived-in space in which they could act out their own intentions (e.g., racing, winning).

Because of physical and timing difficulties, the boys did not enact the expected mathematical pattern. Nonetheless, they tried to make sense of the numbers they saw in the trip. Norman saw that the outcome of the trip was Luke—76, Norman—100, and he was acutely aware of his own difficulties in making the trip. So, Norman immediately hypothesized a new trip: "But I could have probably have went even faster because I, I really, he probably, really, would have been at 72, probably." He realized that if he had gone faster, the net effect would have been to decrease Luke's endpoint on the meterstick. Because of his concern about how the trip might have gone differently, Norman began to describe a mathematical relationship between his and Luke's relative speeds and relative endpoints; this relationship is related to the relationship between rate of change and accumulated distance.

Luke explained the outcome of the trip from his own perspective, saying, "He defeat me," which is consistent with the idea of this trip as a race. Luke went on to describe how Norman was at the 8-cm mark when Luke was at the 4, and at the 12-cm mark when Luke was at the 6. The pattern of accumulated distances that Luke was beginning to describe was caused by the relationship between Luke's and Norman's rates of change, or step sizes. Like Norman, Luke noticed effects of their relative rates of change on the outcome of the trip. Both boys described the mathematical relationships they saw in the activity in terms of the numbers that were meaningful to them in their experiences of the Cuisenaire-rods-and-meterstick environment as a lived-in space.

Number-Table Environment

Norman and Luke used the table environment as a new place to do and interpret a trip with step sizes of 2 and 4. In the table environment, the boys began to pay more attention to number patterns, noticing doubling in the table values for Luke's position, something Luke called "quadrupling up" in the values for Norman's position, even numbers, and "half of something." Norman attempted to describe the relationship between the numbers in the two columns of the table by guessing that he was beating Luke by 50 for the whole trip. However, the constant relationship

between their two positions was not additive, but multiplicative, and closer to Luke's idea: "It's like the half of something." As the boys began to notice their relative positions and step sizes, they often fused these numerical relationships with their own actions: Luke said, *"He's* quadrupling up" and "Twenty-four, *I'll be* at the 12." Norman said, *"I beat him* [italics added] by 50. *I was beating him* [italics added] by 50 the whole time!" The numerical relationships among the boys' speeds and distances became most apparent in the number-table environment. As they explored these relationships, Norman and Luke fused the number relationships with their own actions by describing these numerical relationships in terms of their relative progress in a race. In fusion, the symbol (such as the numeral 12 at a certain position in the number table) and the referent (such as an agent reaching a certain position at a certain time) are merged in one speech act. The boys' fused language helped them to develop the number table into a meaningful lived-in space.

Computer-Software Environment

In the software environment, the boys carried out some actions and intentions similar to those in the previous two environments and some that they had not carried out before. The boys ran their race and determined the result in the software environment, as they had in the previous two environments. However, the boys used the outcome of their trip in the software environment in a new way by comparing it with the outcome of their number-table trip: "[On the computer] he was at 50 and I was at a hundred. And on the paper, he was at 50 and I was on a hundred." In addition, a feature of the software environment allowed the boys to look at their trip in a new way by stepping through it second by second: "We were looking through the time." Norman explored a different sense of rate of change than he had in previous trips, by determining the total time to travel to the end of the 100-unit track: "In 25 seconds, I was at 100." Norman and Luke made the software environment into a lived-in space in which to carry out their own actions and fulfill their own intentions. They used the software environment to check their result from the number table ("We were right!"), to reexamine the trip ("We were looking through the time"), and to find a numerical relationship they had seen in the number table ("'Cause look it was a half [pointing to ending places of runners on screen]"). The boys' use of the software environment to accomplish their own goals allowed them to see a mathematical relationship between the outcome of the trip in the software and the trip in the number table and to discover a new way to see the rate of change of the runners in the software: as total distance traveled in total time.

Norman and Luke populated all three environments with their own actions and intentions, making each environment into a lived-in space by relating it to their own experiences, interests, and intentions. In this way, they explored the mathematical ideas of rate of change, accumulation, and the relationship of these two quantities. The mathematics the boys did was influenced by the intentions they brought to each lived-in space and the kinds of actions they performed in each space.

HOW ARE CONNECTIONS MADE ACROSS ENVIRONMENTS?

In the Cuisenaire-rods-and-meterstick environment, Norman and Luke did not achieve the outcome we had expected. Nonetheless, when they moved on to the table environment, in which they found a number pattern closer to the one we had expected, they still connected their trips in these two environments. The boys described both the Cuisenaire-rods-and-meterstick trip and the number-table trip as if they were races, making a connection to their own experiences with racing. This relationship to racing became a family resemblance the boys used to link their number-table trip and their Cuisenaire-rods-and-meterstick trip.

In the table environment, the boys described number patterns, and they noticed some of the same patterns in their trip in the software environment. Norman noticed the same 50-100 relationship of the ending values in the simulation as he had seen in the number table on paper, and Luke noticed the same *half* relationship between the last two position values in the software that he had noticed for the first few values in the table. Neither of these descriptions is a complete description of one mathematical pattern underlying all trips with step sizes of 2 and 4, but the number patterns the boys noticed in the trips in the software and number-table environments became the family resemblances they used to connect these trips and to build up their sense of what a trip with two different, constant speeds looks like.

The boys connected their trips in all three environments not through the recognition of one core mathematical idea but instead through a web of family resemblances among trips. The boys saw a resemblance between their trips in the first two environments because they both seemed like races in which the boys explored mathematical issues such as who is beating whom and by how much. When the boys moved on to the software environment, they were particularly struck by the similarity between the number patterns in the software and the number-table trips and explored the precise numerical relationships among position values for the two racers. Although the boys did not explore the same mathematical issues in all three trips, we argue that they made use of their experiences of each trip to help build up a family of trips with step sizes of 2 and 4.

If one sees a developing concept as a thread, as Wittgenstein (1958) described it, then "the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres"(p. 32). In these episodes, Norman and Luke developed a family of trips, with step sizes 2 and 4, in which any pair of trips shared some features but no single feature completely defined all the trips. In addition, the unique features of each trip, such as Norman's falling behind in the first trip and the boys' stepping through the time in the last trip, contributed to the boys' overall sense of what a trip can be. Instead of finding a single mathematical definition for a trip, the boys developed family resemblances across trips in all three environments, which helped them to build and broaden their understandings of the mathematics of step size and distance.

PEDAGOGICAL IMPLICATIONS

To understand how the boys made meaning from their work in multiple environments, we have drawn on Wittgenstein's (1958) idea of family resemblances and on Nemirovsky et al.'s (1998) notion of lived-in spaces. We next consider some pedagogical implications of these ideas by looking at two perspectives on teaching students about the meaning of the word *game*, building upon Wittgenstein's description of the meaning and use of this word. On the one hand, a teacher with an essentialist perspective on the meaning of *game* would give great prominence to the definition, because the definition would determine general criteria for which activities belong to the class of games and which do not. The pedagogical approach might be to show examples of games and not-games and to indicate the presence or absence of the defining features in each example. Or one might first give examples and let students discover the definition. In both cases, the definition is the target of the lesson: Once the student correctly applies the definition to categorize activities, one would say that she or he has understood what a game is. On the other hand, if one takes the perspective that developing family resemblances across lived-in spaces is critical to learning, then the most important aspect of learning and knowing what a game is becomes one's life experience with games. On the basis of the kinds of games one has participated in or watched and one's impressions of these games, one recognizes certain activities as being of the family of games and others as not of this family. On that same basis, one assesses whether some definitions of *game* are better than others. From this point of view, the pedagogical focus is on allowing and encouraging students to participate in a range of games, to make these games into lived-in spaces for themselves, and to reflect on their experiences of playing games, through discussing, symbolizing, and comparing games.

To describe how the development of family resemblances across multiple lived-in spaces can contribute to our own mathematical understandings, we reflect on how one understands the familiar idea of one half. Try out these two tasks yourself. First, find one half of the quantity 3275 and write your answer. Next, walk across a room once, and then walk across the room again but try to walk half as fast as you did the first time. These two examples clearly share the mathematical idea of dividing something in half, but probably your experiences of dividing something in half had many differences as well as similarities. Dividing 3275 in half most likely drew on your computational ability and your number sense, causing you to think about one half as a relation between two numbers. Walking across the room half as fast as before probably caused you to think about how to qualitatively compare one or more of the quantities speed, distance, and time when comparing your two motions across the room. You may also have tried to feel the quantities of speed or time in your body as you made the motion itself. In these ways you may have related two experiences of speed to arrive at one half, or you may have perceived the continuous quantities of time or speed and compared them.

Although one may be tempted to look for the essential element of "halfness" in each of these experiences and to try to find ways to give students access to this element, one's own sense of halfness comes from these experiences and many others like them. The similarities and the differences among these experiences of *one half* and all the experiences of *one-half* relationships one has had in a lifetime constitute the family of *one half* that allows one to recognize what are *one-half* relationships and what are not. Being a proficient user of the concept of *one half* is being able to move fluently through extended networks of similarities and differences among multiple experiences of *one-half* relationships. Analogously, the learning that takes place in Norman's and Luke's experiences with trips in three environments cannot be accounted for in terms of some ultimate mathematical definition abstracted from their work. Instead, the boys engaged in a process of making each environment into a lived-in space for themselves and of developing family resemblances across their trips in these lived-in spaces, to create a family of trips that included the similarities among trips as well as the distinct identity of each trip.

Of course it is sometimes important for a teacher and her or his students to step back from a diverse set of activities and ask what they have in common and to reflect on the general mathematical principles that describe the activities. However, we argue that these general principles become meaningful and relevant only to the extent that they are rooted in an ongoing background of experiences.

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